UNCLASSIFIED

AD 409 172

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

N-63-4-2

SELENIA S.p.A.

AFCRL - 63 - 642

AD

AF 61(052)-145
TN-3

29 January 1963

CATALOGED BY DDC
AS AD No. 409172

TECHNICAL NOTE No. 3

PROPAGATION OF WAVES IN INHOMOGENEOUS MAGNETO-ACTIVE PLASMA COLUMNS

Ing. Pietro de SANTIS

SOCIETA' SELENIA S.p.A.
VIA TIBURTINA KM 12,4
R O M A

The research reported in this document has been sponsored by the Cambridge Laboratories, OAR through the European Office, Aerospace Research, United States Air Force.

409172

DDC
RECEIVED
JUL 12 1963
TISIA B

SELENIA S.p.A.

ABSTRACT

A preliminary theoretical investigation of the propagation of a quasi-static E mode in a plasma column with non-uniform electron density distribution in the cross-section, has already been performed in the special case of a density independent $\varepsilon_3/\varepsilon_1$ ratio (Technical Summary Report N.2).

Starting from Maxwell's equations, a more general case is here considered, where a parabolic distribution has been taken as an analytical model for the electron density profile in the cross - section.

Electron density is supposed to vanish at a point outside the plasma column ; in the limit for the zero point to go to infinity, uniform case equations are obtained.

Letting the metal walls move to infinity, or coincide with the plasma boundary, the completely filled waveguide and the plasma column in free space dispersion curves have been verified.

The Ritz variational method is used to minimize the error over the cross section when an approximation function for the field distribution in the plasma is assumed.

TABLE OF CONTENTS

1.- Introduction .....pag. 2

2.- Derivation of the equation for $E_z(u)$ .....pag. 5

3.- Solution of equation for $E_z(u)$ inside the plasma .....pag. 7

4.- Solution of equation for $E_z(u)$ outside the plasma .....pag. 8

5.- Derivation of the dispersion solution .....pag. 8

6.- Particular cases .....pag. 10

7.- Comparison between the approximate solution and the exact solution .....pag. 13

8.- Conclusions .....pag. 14

9.- Bibliography .....pag. 16

1. Introdution

For analitical simplicity let us consider a plane geometry instead of a cylindrical one and let us suppose that a slab of inhomogeneous gyrotropic plasma is located between two parallel lossless metal plates, the distance between them being 2D and the slab thickness being 2d.

The coordinate reference frame is chosen so that the slab is perpendicular to the x axis with its surfaces at $\pm$ d, and the z axis in the direction of propagating waves.

The magnetic field B points along the plates in the direction of the z axis. The plasma slab is inhomogeneous along the x axis and infinite in the y direction.

The field components and the electron density distribution function are expressed in terms of the independent variable $u = \beta x$, β being the longitudinal propagation constant.

The whole structure and the cross sectional distributions assumed for the E_z component and for the electron density are shown in fig. 1.

The analysis proceeds by calculating the longitudinal components of the electric field inside and outside the slab. Substituting these components into the boundary conditions then leads to a dispersion relation, whose solutions are given for the cases shown in fig. 2.

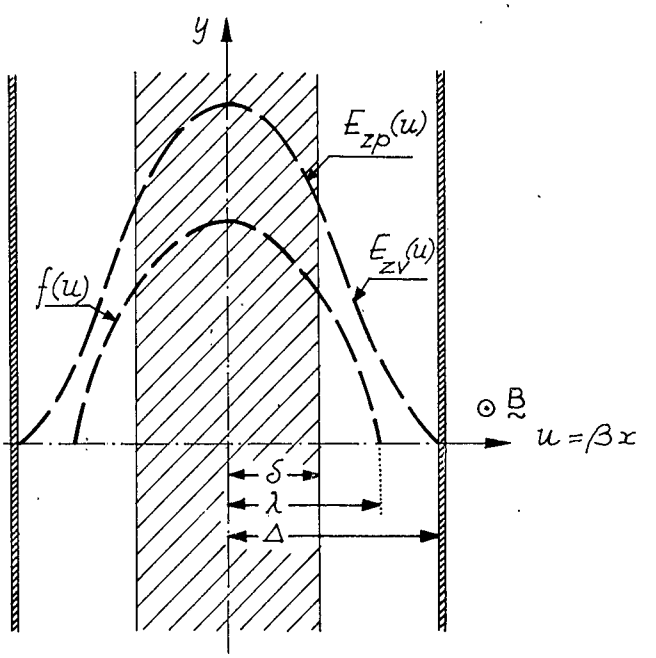


Fig. 1

The following symbols are used :

$E_{zp}(u)$: cross sectional distribution of the longitudinal electric field inside the plasma.

$E_{zv}(u)$: cross sectional distribution of the longitudinal electric field outside the plasma.

E_0, E_1 : amplitude constants of $E_{zp}(u)$.

β : longitudinal propagation constant

$u = \beta x$

$2d$: slab thickness

$2D$: distance between plates.

l : value of x at which the electron density is zero.

$\delta = \beta d$

$\Delta = \beta D$

$\lambda = \beta l$

$f(u)$: cross sectional distribution of the electron density.

L : second order differential linear operator.

μ : parameter characterizing the cross-sectional inhomogeneity.

ω_b : cyclotron angular frequency.

ω_p : plasma angular frequency at the center of the slab.

$b_0 = \omega_p^2/\omega^2$

$\alpha^2 = \omega_b^2/\omega^2$

$\tau = 1 - (1 - \alpha^2)/b_0$

$\tau_0 = 1 - 1/b_0$

2. Derivation of the equation for E_z (u).

Starting from Maxwell's equations a wave equation for $\underset{\sim}{E} \equiv (E_x\ E_y\ E_z)$ is obtained :

$$\nabla^2 \underset{\sim}{E} - \nabla \nabla \cdot \underset{\sim}{E} + \omega^2 \mu_0 \overleftrightarrow{\varepsilon} \cdot \underset{\sim}{E} = 0 \qquad (1)$$

where $\overleftrightarrow{\varepsilon}$ is the dyadic absolute permittivity.

If quasi-neutrality is assumed,(dielectric approximation) :

$$\nabla \cdot \left(\overleftrightarrow{\varepsilon} \cdot \underset{\sim}{E} \right) = 0 \qquad (2)$$

and, hence, $\nabla \cdot \underset{\sim}{E}$ is derived, the z component of eq.(1), in the quasi-static approximation, can be written in the form :

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{1}{\varepsilon_1} \frac{\partial \varepsilon_1}{\partial x} \frac{\partial E_z}{\partial x} - \beta^2 \frac{\varepsilon_3}{\varepsilon_1} E_z = 0 \qquad (3)$$

where ε_1 and ε_3 are the diagonal components of $\overleftrightarrow{\varepsilon}$ and an exp j ($\omega t - \beta z$) dependence is assumed.

In eq.(3) the second term is due to the inhomogeneity in the cross-section and the absence of any component of the magnetic field is due to the quasi-static approximation.

If the normalized variable $u = \beta x$ is used eq.(3) can be written :

$$\frac{\partial^2 E_z}{\partial u^2} + \frac{1}{\varepsilon_1} \frac{\partial \varepsilon_1}{\partial u} \frac{\partial E_z}{\partial u} - \frac{\varepsilon_3}{\varepsilon_1} E_z = 0 \qquad (4)$$

The following distributions in the cross section (plasma region) are now assumed :

$$\varepsilon_1(u) = 1 - \frac{b(u)}{1-\alpha^2}$$

$$\varepsilon_3(u) = 1 - b(u)$$

with $\alpha^2 = \frac{\omega_b^2}{\omega^2}$ and $b(u) = b_0 f(u) = (\omega_p^2/\omega^2) f(u)$, where $f(u)$ specifies a particular kind of density distribution.

If

$$f(u) = 1 - \frac{u^2}{\lambda^2}$$

where

$$\lambda^2 = \beta^2 l^2$$

$\varepsilon_1(u)$ and $\varepsilon_3(u)$ can be rewritten

$$\varepsilon_1(u) = 1 - \frac{b_0(\lambda^2 - u^2)}{\lambda^2(1-\alpha^2)}$$

$$\varepsilon_3(u) = 1 - b_0\left(1 - \frac{u^2}{\lambda^2}\right)$$

Substituting these values into eq.(4) the following equation is obtained :

$$L(E_z) = (u^2 - \lambda^2\tau)\frac{\partial^2 E_z}{\partial u^2} + 2u\frac{\partial E_z}{\partial u} + (\lambda^2\tau_0 - u^2)(1-\alpha^2)E_z = 0 \qquad (5)$$

where

$$\tau = 1 - \frac{1-\alpha^2}{b_0}$$

$$\tau_0 = 1 - \frac{1}{b_0}$$

and the first member of the equation has been designated with the operational symbol L (E_z).

3. Solution of the equation for $E_z(u)$ inside the plasma.

A solution of eq. (5) in the plasma region can be obtained with the aid of the Ritz method by assuming a test function as an admissible analytical expression for $E_z(u)$.

If a linear combination of suitable functions is chosen for $E_{zp}(u)$,

$$E_{zp}(u) = E_o + \Sigma_k a_k E_k(u)$$

a first order approximation is given by

$$E_{zp}(u) = E_o + E_1 (\delta^2 - u^2) \tag{6}$$

where $$\delta = \beta d$$

Introducing the expression (6) of $E_{zp}(u)$ in eq. (5) the problem is reduced to a "minimum" problem with the following boundary conditions :

$$\begin{aligned} & E_{zp}(\delta) = E_{zv}(\delta) \\ & \varepsilon_1(\delta) E_{xp}(\delta) = E_{xv}(\delta) \\ & E_{zv}(\Delta) = 0 \end{aligned} \tag{7}$$

The problem can be mathematically formulated as follows:

$$E_1 \int_0^{\delta} (\delta^2 - u^2) L (\delta^2 - u^2) \, du = - E_o \int_0^{\delta} (\delta^2 - u^2)(\lambda^2 \tau_o - u^2)(1 - \alpha^2) \, du \tag{8}$$

and if the degree of homogeneity in the cross-section is specified by a characteristic parameter μ defined as follows :

$$\mu = \frac{\lambda}{\delta}$$

eq.(8) can be reduced to :

$$E_1\left[\delta^5\left(\frac{2}{3}\tau\mu^2-\frac{2}{5}\right)+\left(1-\alpha^2\right)\left(\frac{4}{15}\mu^2\tau_0-\frac{4}{105}\right)\delta^7\right]= \tag{9}$$

$$=-E_0\delta^5\left(1-\alpha^2\right)\left(\frac{\mu^2\tau_0}{3}-\frac{1}{15}\right)$$

4. Solution of equation for $E_z(u)$ outside the plasma.

Outside the plasma eq.(5) reduces to the following ordinary second order differential equation :

$$\frac{\partial^2 E_z}{\partial u^2}-E_z=0 \tag{10}$$

By imposing the boundary condition at $u=\Delta$, the solution of eq. (10) is :

$$E_{zv}(u)=A\left(e^{-u}-e^{-2\Delta+u}\right) \tag{11}$$

where A is an arbitrary amplitude constant.

5. Derivation of the dispersion relation.

The eigen value δ can be calculated by imposing the boundary conditions (7), at $u=\delta$.

When this is done, the following two equations are obtained :

$$E_0=A\left(e^{-\delta}-e^{-2\Delta+\delta}\right) \tag{12}$$

$$E_1=\frac{A}{2\delta\mathcal{E}_1(\delta)}\left(e^{-\delta}+e^{-2\Delta+\delta}\right)$$

Substituting these expressions into eq.(9), we have :

$$\left(\frac{\tau\mu^2}{3}-\frac{1}{5}\right)+\left(1-\alpha^2\right)\left(\frac{2}{15}\tau_0\mu^2-\frac{2}{105}\right)\delta^2 =$$

$$= -\delta\, th\left(\Delta-\delta\right)\left(1-\alpha^2\right)\left(\frac{\mu^2\tau_0}{3}-\frac{1}{15}\right)\left[1-\frac{b_0\left(\mu^2-1\right)}{\mu^2\left(1-\alpha^2\right)}\right] \quad (13)$$

This is the dispersion relation in the general case of $\delta<\lambda<\infty$ and $\delta<\Delta<\infty$. Some particular cases of special interest can be derived from this; they are shown in the following table.

PHYSICAL STRUCTURE	DENSITY DISTRIBUTION	CURVE DESIGNATION
a) $\Delta=\delta$ (COMPLETELY FILLED WAVEGUIDE)	$\mu=1$ (PARABOLIC)	a_1
	$\mu=\infty$ (UNIFORM)	a_∞
b) $\Delta\to\infty$ (SLAB IN FREE SPACE)	$\mu=1$ (PARABOLIC)	b_1
	$1<\mu<\infty$ (PARABOLIC) (FOR EX. $\mu^2=2$)	b_2
	$\mu=\infty$ (UNIFORM)	b_∞

fig. 2

6. Particular cases.

In the a) case if we insert $\Delta = \delta$ in eq. 13) the R.H.S vanishes and the following equation is left :

$$\left(\frac{I\mu^2}{3} - \frac{1}{5}\right) + \left(1-\alpha^2\right)\left(\frac{2}{15}\tau_0\mu^2 - \frac{2}{105}\right)\delta^2 = 0 \qquad (14)$$

or the alternative form :

$$\left[\frac{\mu^2}{3}\left(1 - \frac{1-\alpha^2}{b_0}\right) - \frac{1}{5}\right] + \left(1-\alpha^2\right)\left[\frac{2\mu^2}{15}\left(1-\frac{1}{b_0}\right) - \frac{2}{105}\right]\delta^2 = 0 \qquad (15)$$

If $\mu = 1$ the above equation reduces to

$$\delta^2 = -\frac{7/6}{1-\alpha^2}\;\frac{b_0 - 5/2\left(1-\alpha^2\right)}{b_0 - 7/6} \qquad (16)$$

which is the dispersion equation of a completely filled waveguide with a parabolic electron density distribution in the cross-section.

A plot of eq. (16) in the plane $\bar{\omega}^2 = \frac{\omega^2}{\omega_p^2} = \frac{1}{b_0}$ vs. $|\delta|$ yields the curves designated by a_1. (Each curve has been calculated for a fixed value of the parameter $a = \frac{\omega_b^2}{\omega_p^2} = \frac{\alpha^2}{b_0}$ (see tables I and I').

If $\mu = \infty$ eq. (15) becomes :

$$\frac{1}{3}\left(1 - \frac{1-\alpha^2}{b_0}\right) + \left(1-\alpha^2\right)\left[\frac{2}{15}\left(1-\frac{1}{b_0}\right)\right] = 0 \qquad (17)$$

If $\alpha = 0$ (isotropic plasma)

$$\delta^2 = -\frac{5}{2}$$

there is no propagation.

If $\alpha \neq 0$ (gyrotropic plasma)

$$\delta^2 = -\frac{5/2}{1-\alpha^2}\,\frac{b_0-(1-\alpha^2)}{b_0-1} \tag{18}$$

which is the dispersion relation of a completely filled waveguide with a uniform distribution of the electron density in the cross-section.

If eq. (18) is plotted in the $\bar{\omega}^2$-$|\delta|$ plane, the curves a_∞ are obtained. (Tables I, I').

In the b) case if we let $\Delta \to \infty$ in eq. (13), this eq. reduces to+:

$$\left(\frac{\tau\mu^2}{3}-\frac{1}{5}\right)+\left(1-\alpha^2\right)\left(\frac{2\tau_0\mu^2}{15}-\frac{2}{105}\right)\delta^2 = -|\delta|\left(1-\alpha^2\right)\left(\frac{\tau_0\mu^2}{3}-\frac{1}{15}\right)\left[1-\frac{b_0(\mu^2-1)}{\mu^2(\alpha^2-1)}\right] \tag{19}$$

which in terms of the normalized $\bar{\omega}^2$, has the form;

+ When Δ tends to infinity our physical structure reduces to a slab of plasma placed in an unlimited space. In this case th $(\Delta-\delta) = 1$ and the absolute value of δ must be considered, if reciprocity is to be respected.
From a mathematical stand point it is to be noticed that when $\beta<0$ the limit

$$\lim_{\Pi\to\infty} \beta d \,\mathrm{th}\,\beta(\Pi-d)$$

tends to $|\beta|d$ in the same manner as when $\beta>0$.

For $a = \frac{1}{2}$

$$\left[\left(\frac{2\mu^2}{15} - \frac{2}{105}\right) - \frac{2\mu^2}{15}\bar{\omega}^2\right]\delta^2 + \left[\frac{\mu^2}{3} - \frac{1}{15} - \frac{\mu^2}{3}\bar{\omega}^2\right] \left[1 - \frac{\mu^2 - 1}{\mu^2\left(\bar{\omega}^2 - \frac{1}{2}\right)}\right]|\delta| + 2\bar{\omega}^2\left[\frac{\mu^2/3 - 1/5}{2\bar{\omega}^2 - 1} - \frac{\mu^2}{6}\right] = 0 \tag{20}$$

For $a = 2$

$$\left[\frac{2\mu^2}{15} - \frac{2}{105} - \frac{2\mu^2\bar{\omega}^2}{15}\right]\delta^2 + \left[\frac{\mu^2}{3} - \frac{1}{15} - \frac{\mu^2\bar{\omega}^2}{3}\right] \left[1 - \frac{\mu^2 - 1}{\mu^2\left(\bar{\omega}^2 - 2\right)}\right]|\delta| + \bar{\omega}^2\left[\frac{\mu^2/3 - 1/5}{\bar{\omega}^2 - 2} - \frac{\mu^2}{3}\right] = 0 \tag{20'}$$

If $\mu^2 = 1$ eq. 20, 20' reduce to :

$$\left(\frac{12}{105} - \frac{2}{15}\bar{\omega}^2\right)\delta^2 + \left(\frac{4}{15} - \frac{1}{3}\bar{\omega}^2\right)|\delta| + \bar{\omega}^2\left(\frac{4/15}{2\bar{\omega}^2 - 1} - \frac{1}{3}\right) = 0 \tag{21}$$

$$\delta^2\left(\frac{12}{35} - \frac{2}{5}\bar{\omega}^2\right) + \left(\frac{4}{5} - \bar{\omega}^2\right)|\delta| + \bar{\omega}^2\left(\frac{2}{5(\bar{\omega}^2 - 2)} - 1\right) = 0 \tag{21'}$$

Eqs. (21) are the dispersion relations for a slab of plasma in free space with a parabolic electron density distribution in the cross section. (curves b_4 tables II, II').

If $\mu = \infty$, eqs 20 reduce to :

$$\frac{2}{5}\left(1 - \bar{\omega}^2\right)\delta^2 + \left(1 - \bar{\omega}^2\right)\frac{2\bar{\omega}^2 - 3}{2\bar{\omega}^2 - 1}|\delta| + \bar{\omega}^2\,\frac{3 - 2\bar{\omega}^2}{2\bar{\omega}^2 - 1} = 0 \tag{22}$$

$$\frac{2}{5}\left(1 - \bar{\omega}^2\right)\delta^2 + \left(1 - \bar{\omega}^2\right)\frac{\bar{\omega}^2 - 3}{\bar{\omega}^2 - 2}|\delta| + \bar{\omega}^2\,\frac{3 - \bar{\omega}^2}{\bar{\omega}^2 - 2} = 0 \tag{22'}$$

which are the dispersion relations for a slab of plasma in free space with a uniform electron density distribution in the cross-section. (Curves b_∞; tables II, II'). Dispersion curves, for the case $\mu^2 = 2$ have also been calculated and plotted (curves b_2).

7. Comparison between the approximate solution and the exact solution.

A direct analysis of the admissible propagation modes in a slab of uniform plasma in free space shows that a cosinus distribution of E_z in the cross-section is possible when a quasi static E mode is considered.

Imposition of boundary conditions yields a characteristic equation and a dispersion relation which is

$$\varepsilon_1 \sqrt{-\frac{\varepsilon_3}{\varepsilon_1}}\ tg\left(\sqrt{-\frac{\varepsilon_3}{\varepsilon_1}}\,|\delta|\right) = 1 \qquad (23)$$

where the modulus of $|\delta|$ has been chosen in accordance with reciprocity principle.

In terms of the new normalized variable $\bar{\omega}^2$ eq. (23) can be rewritten :

$$tg\left[|\delta|\sqrt{\frac{(1-\bar{\omega}^2)(\bar{\omega}^2-2)}{\bar{\omega}^2(\bar{\omega}^2-3)}}\right] = \frac{\bar{\omega}^2-2}{\bar{\omega}^2-3}\left[\sqrt{\frac{\bar{\omega}^2(\bar{\omega}^2-3)}{(1-\bar{\omega}^2)(\bar{\omega}^2-2)}}\right] \qquad (24)$$

This equation can be solved numerically and dispersion curves can be derived and compared with curve b_∞.(Table IV, IV'). (A subscript "e" is used to indicate "exact solution" curves).

CONCLUSIONS

Application of a variational method to a quasi-static problem, has been made. The choice of the test function, based on the knowledge of the field distribution in the uniform plasma case, represents a first order mathematical approximation capable of describing a mode with the lowest order transversal distribution.

An inspection of the dispersion curves, shows that, within the above limits :

1. - passing from cases a_∞ and b_∞ (uniform electron density distribution) to cases a_1 and b_1 (parabolic electron density distribution) the upper band shrinks.
2. - For the a_1 curve, when the parameter $a = \left(\frac{\omega_b}{\omega_p}\right)^2$ equals the critical value $a_c = 0{,}456$ the upper band shrinks to zero and if $a < a_c$, propagation occurs only in the pass band $0 < \bar{\omega}^2 < a$

   For the b curves (slab of plasma in free space) critical values of a have been computed for different μ^2 and the results plotted in fig. 3.

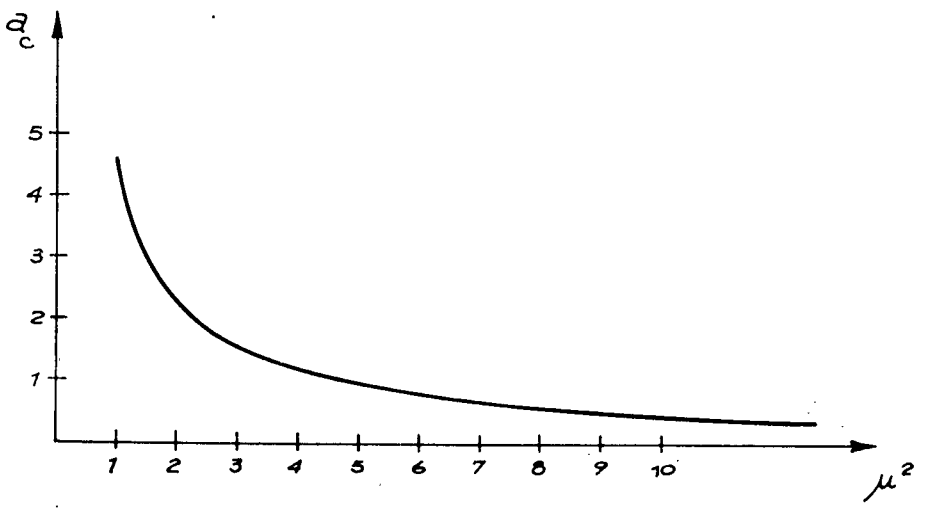


fig. 3

3. - The asymptotic $\bar{\omega}^2$ value, equal to 1 in the uniform case, in the passage from a_∞ to a_1 case, decreases whereas the asymptotic value equal to a remains unchanged. This is expected, since the average electron density, in the passage from a_∞ to a_1 case, decreases as well.

4. - Letting the metal walls move away from the plasma boundary does not affect the behaviour of the dispersion curves as much as varying the electron distribution in the cross-section.(Table III and III').

5. - A comparative exam of the dispersion curves obtained by variational methods and by exact mathematical analysis, for a slab of plasma in free space, shows a fairly good agreement in the upper band, and an excellent agreement in the lower band when $\omega_b > \omega_p$ (Table IV'). When $\omega_b < \omega_p$ the lower branches of the curves show a different behaviour. (Table IV).

It has, in fact, been demonstrated that, with or without quasi static approximation, (see ref. 5) the lowest order mode dispersion curve approaches the asymptotic value $\sqrt{\frac{\omega_p^2 + \omega_b^2}{2}}$ whenever $\omega_p > \omega_b$ whereas application of the variational method provides only an asymptotic value for $\omega = \omega_b$.

BIBLIOGRAPHY

All the references in the previous reports, are to be included , and :

1. - COLLATZ - Numerical treatment of differential equations. Springer Verlag - 1960

2. - PICONE - Lezioni di Analisi Funzionale - Tumminelli - 1947- Cap.IV

3. - KISLOV-BOGDANOV - Interaction between slow plasma waves and an electron stream.
Proceedings of the symposium on electromagnetics and fluid dynamics of gaseous plasma.
New York - April 1961 - Pag. 249 - 268

4. - POSTNOV - Radio Engineering and Electronics - 10 - 63 - 1960

5. - FORMATO & GILARDINI - Propagation characteristics of magneto-ionic plasma columns. - Journal of Research of the National Bureau of Standards.
D. Radio Propagation - Sept. - Oct. 1962

6. - TRIVELPIECE - Space charge waves and plasma diagnostics.
Engineering aspects of magnetohydrodynamics - Columbia University Press. 1962 - Pages 419-437-

7. - BEVC-EVERHART - Fast-wave propagation in plasma filled waveguides.
Journal of Electronics and Control. Sept. 1962.

$\frac{\omega_b^2}{\omega_p^2} = 0.5$

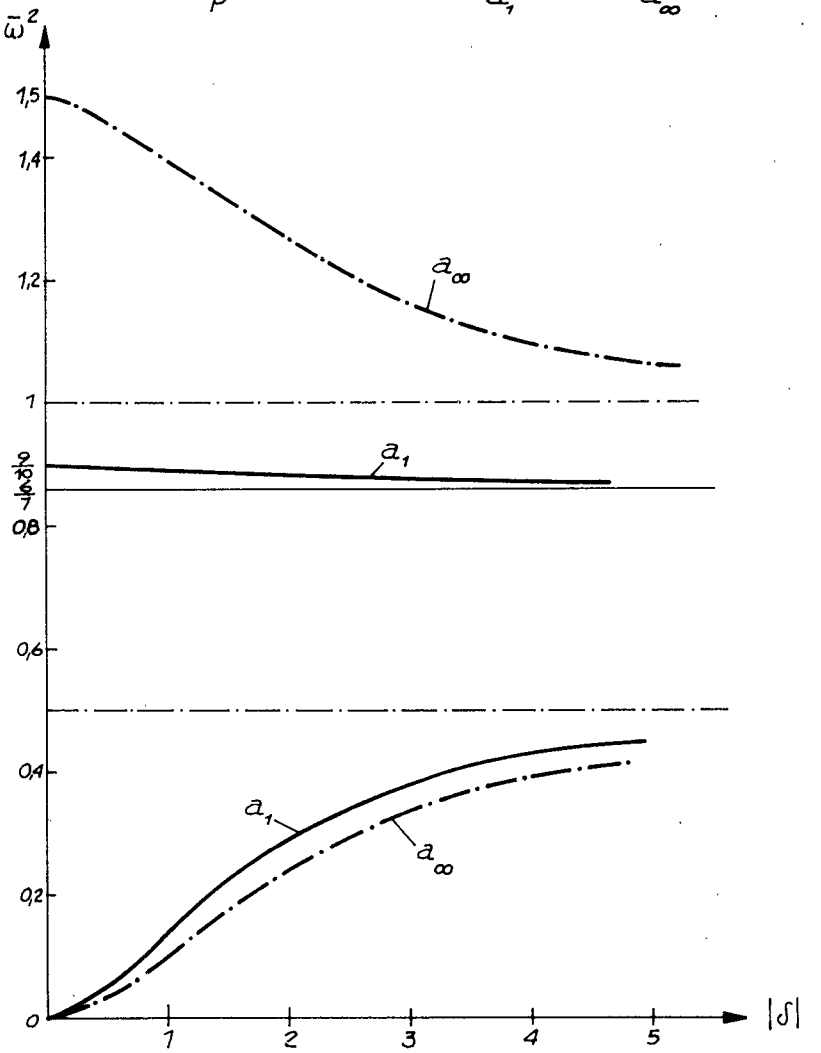


TABLE I - CURVES a_1 , a_∞

$$\frac{\omega_b^2}{\omega_p^2} = 2$$

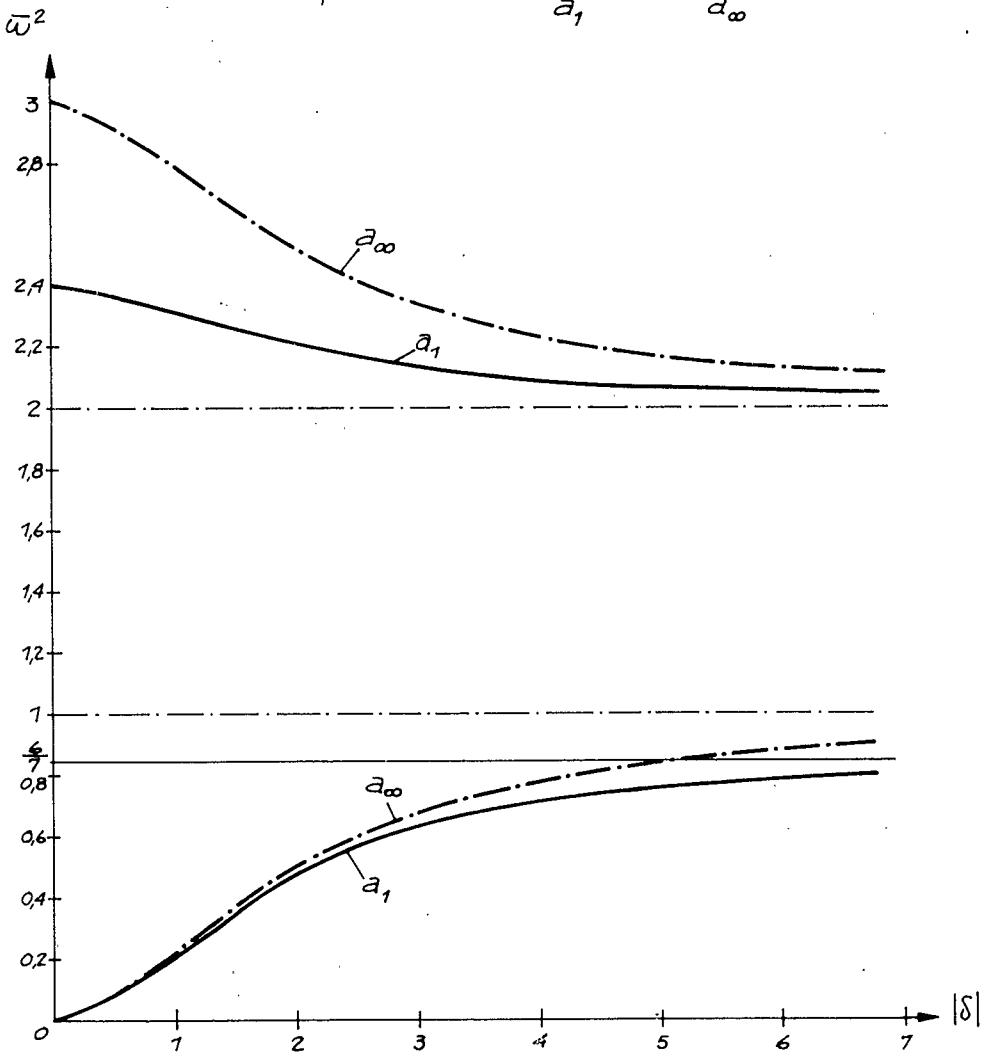
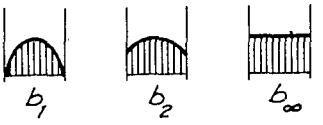


TABLE I' - CURVES a_1 , a_∞

$$\frac{\omega_b^2}{\omega_p^2} = \frac{1}{2}.$$



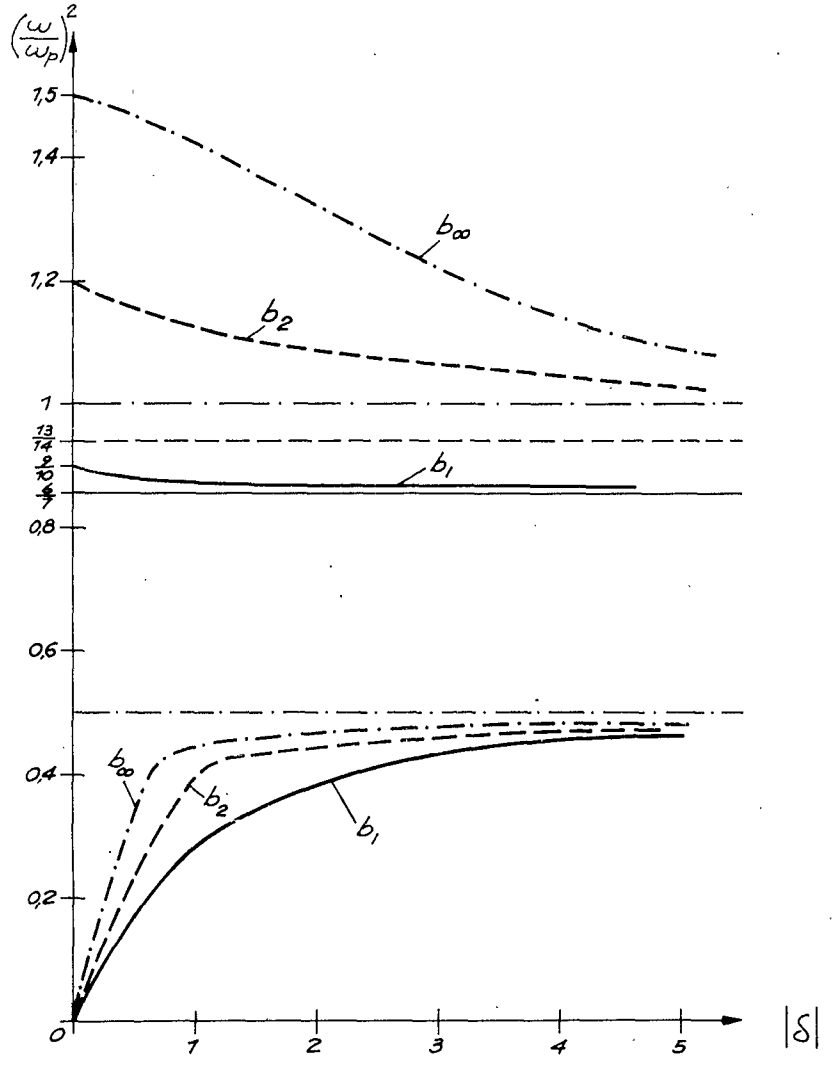
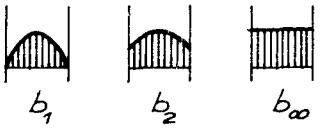


TABLE II - CURVES b_1, b_2, b_∞

$$\frac{\omega_b^2}{\omega_p^2} = 2$$



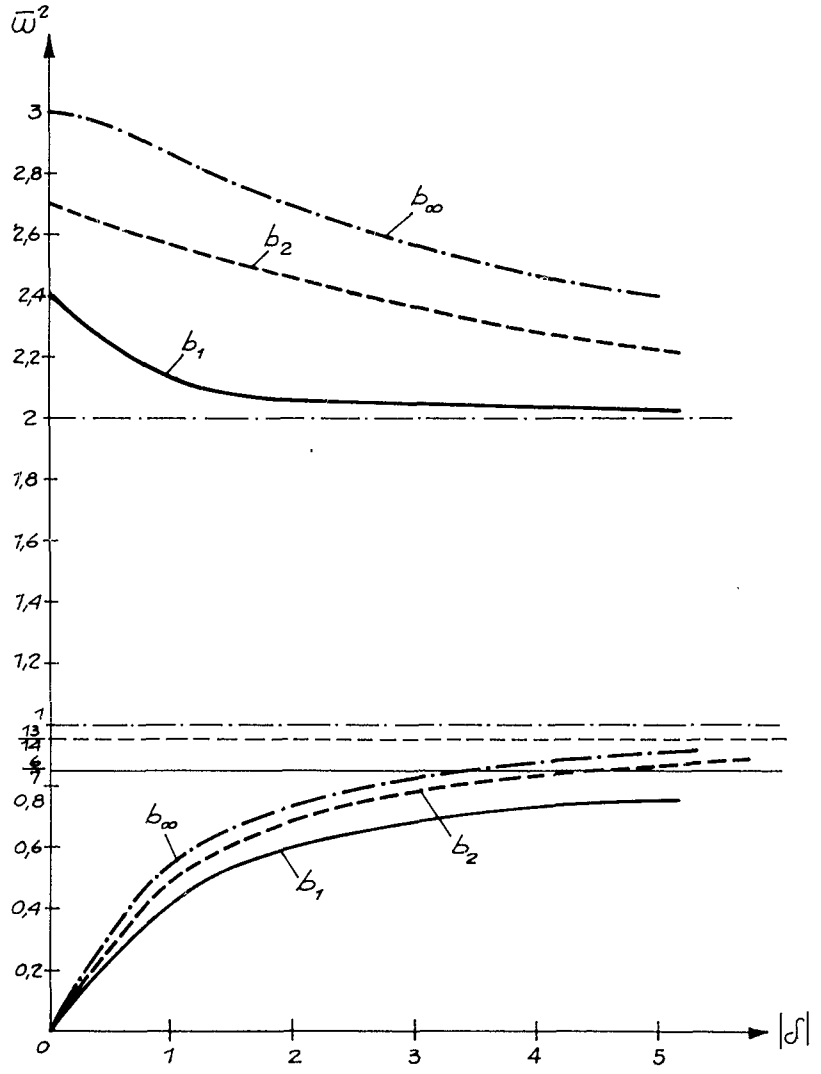


TABLE II' - CURVES b_1, b_2, b_∞

SELENIA S.p.A.

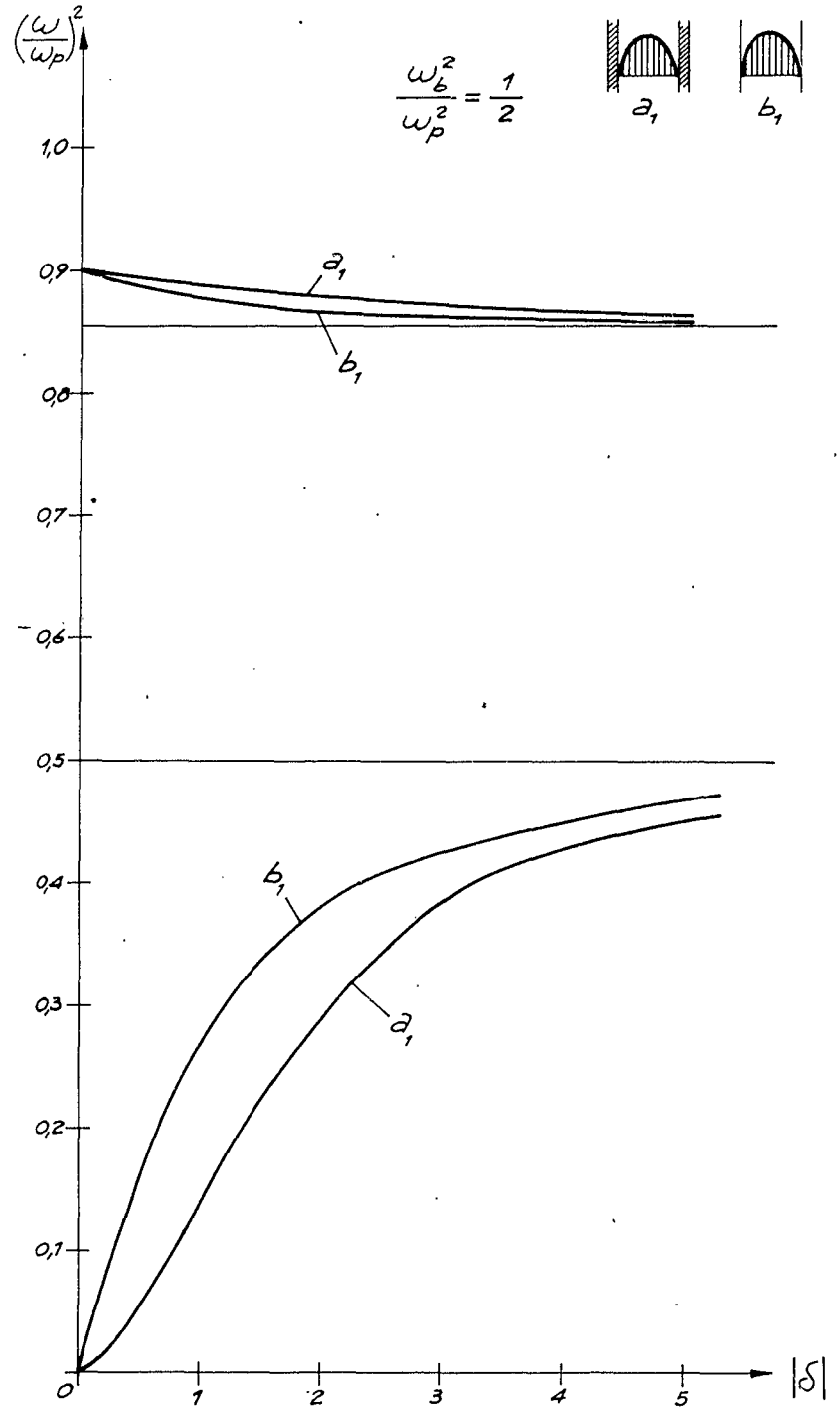


TABLE III - CURVES a_1, b_1

$$\frac{\omega_b^2}{\omega_p^2} = 2$$

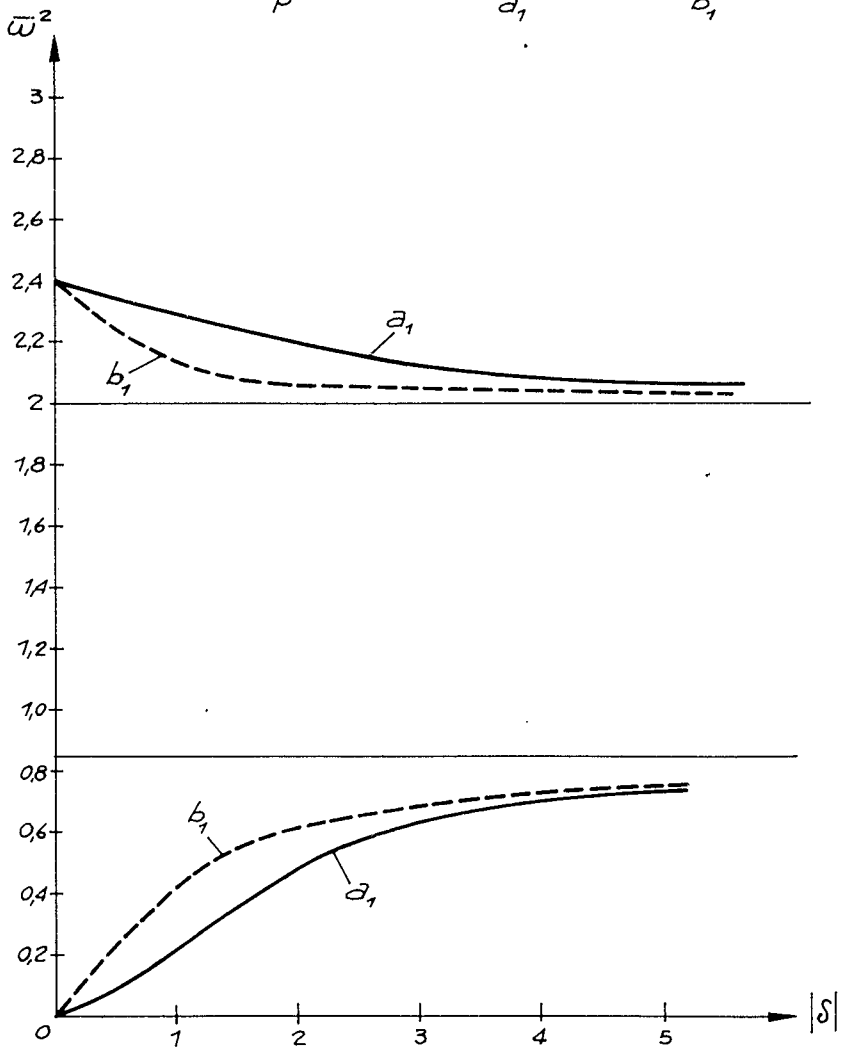


TABLE III' - CURVES a_1 , b_1

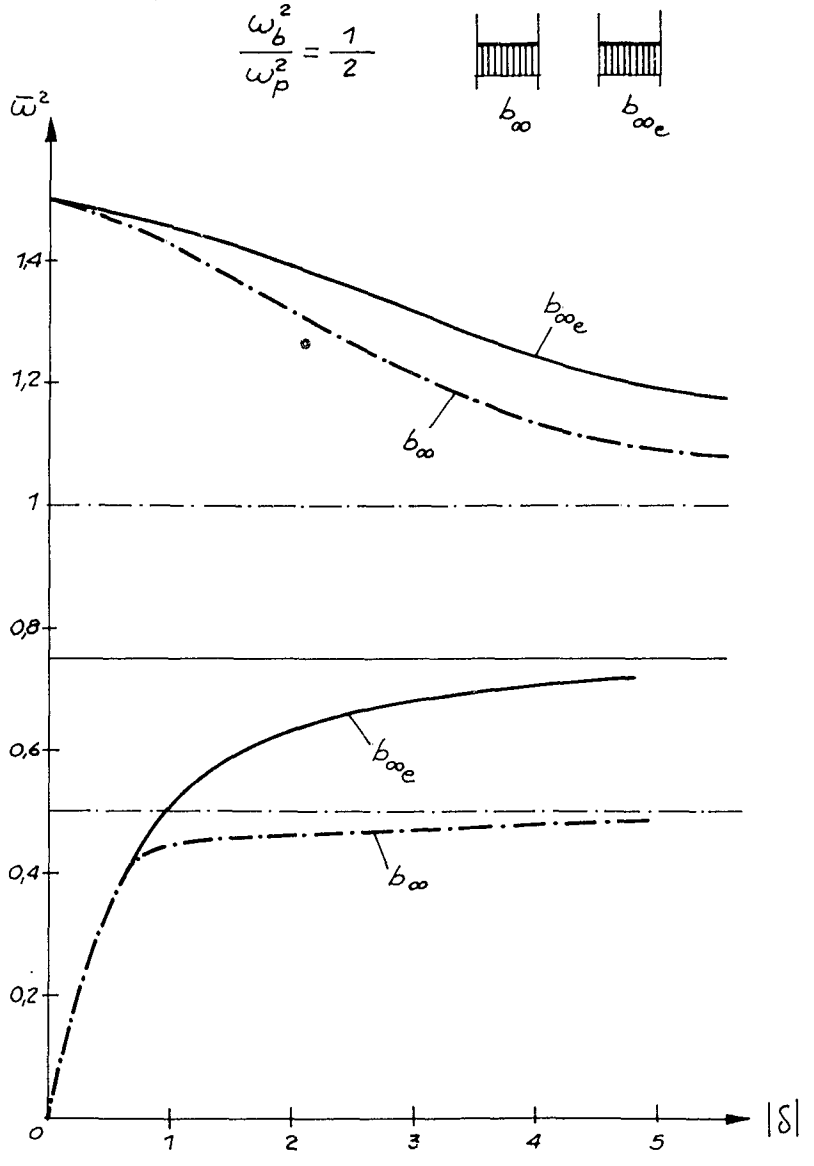


TABLE IV - CURVES b_∞ , $b_{\infty e}$

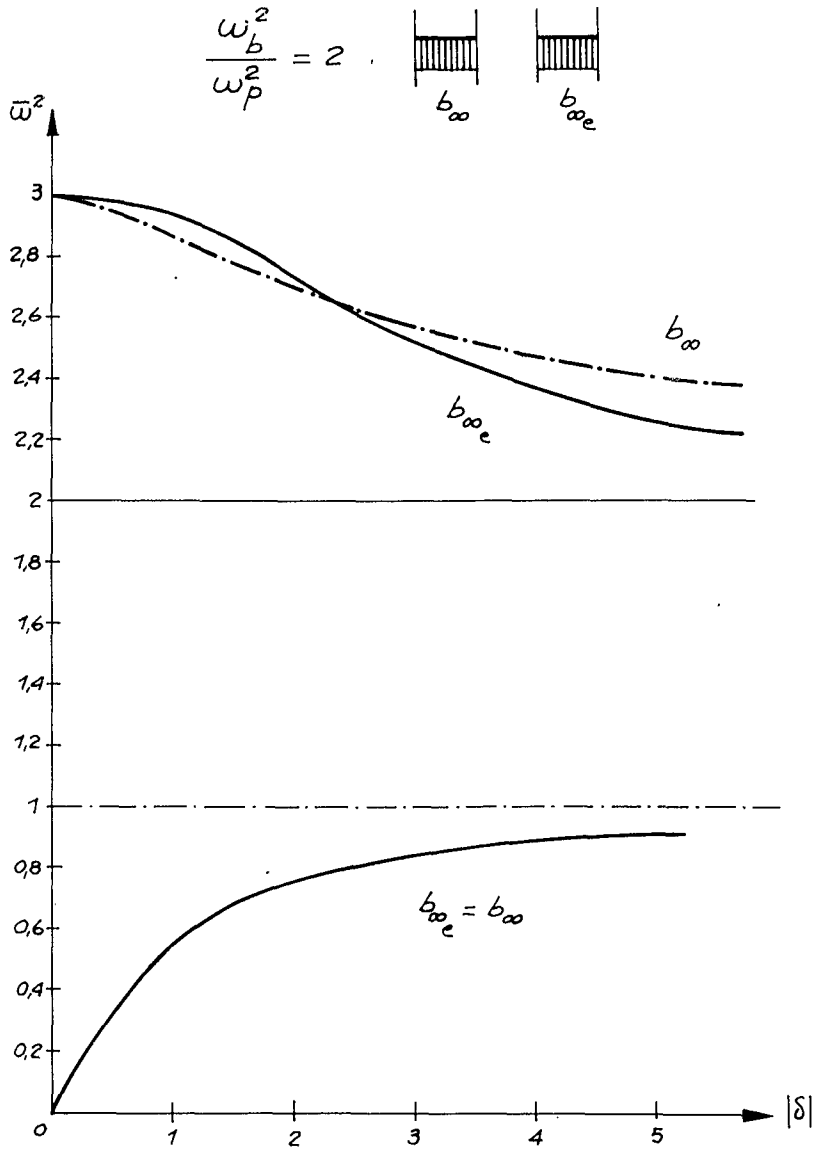


TABLE IV' - CURVES b_∞ , b_{∞_e}